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FREE CONVECTION BOUNDARY LAYERS ON AXI-SYMMETRIC AND TWO-DIMENSIONAL BODIES OF ARBITRARY SHAPE IN A SATURATED POROUS MEDIUM

J. H. MERKIN
School of Mathematics, University of Leeds, Leeds LS2 9JT, England

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NOMENCLATURE

g, acceleration due to gravity;
k, thermal conductivity of the porous medium;
K, permeability of the porous medium;
l, typical length scale of the body;
Q, local heat flux;
$r(x)$, radius of the (axi-symmetrical) body;
T, temperature;
T_0, temperature of the ambient fluid;
T_w, temperature on the surface of the body;
ΔT, temperature difference, $= T_w - T_0$;
u, Darcy's law velocity in the x-direction;
v, Darcy's law velocity in y-direction;
x, coordinate along the body surface;
y, coordinate normal to the body surface.

Greek symbols

α, equivalent thermal diffusivity;
β, coefficient of thermal expansion;
μ, viscosity of the convective fluid;
ρ, density of the convective fluid;
ϕ, angle between the normal drawn outwards from the body and the downward vertical.

BOUNDARY layers on bodies immersed in saturated porous media for both free and mixed convection have been the subject of several recent papers [1–8]. These studies have been limited to bodies with a simple geometric configuration. Here we consider the free convection boundary layers on two-dimensional and axi-symmetric bodies of arbitrary shape embedded in a saturated porous medium. The bodies are assumed to be impermeable and at a constant temperature different to that of the surrounding fluid. We show that the governing equations possess a similarity solution for any body shape, with a resulting ordinary differential equation which has been solved previously by Ackroyd [9], though in an entirely different context. This is not the usual case for free convection boundary layers where similarity solutions are possible only for a limited class of body shapes, [10].

Two types of body are discussed. Firstly we consider an infinite cylinder mounted with its generators horizontal so that the flow is two-dimensional round the cylinder. Here we use the coordinate x to measure distance round the cylinder from the lowest point. Secondly we consider an axi-symmetric body mounted with its axis of symmetry vertical. Here we use the coordinate x to measure distance along the body surface from the lowest point. In both cases the coordinate y measures distance normal to the body and $\phi(x)$ is the angle between the outward normal and the downward vertical. The coordinate system is shown in Fig. 1.

The boundary-layer equations are, following Wooding [11],

$$u = \frac{g\beta K\rho}{\mu}(T - T_0)\sin\phi \tag{1}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial T}{\partial y^2} \tag{2}$$

$$\frac{\partial}{\partial x}(r^n u) + \frac{\partial}{\partial y}(r^n v) = 0 \tag{3}$$

where $n = 0$ for two-dimensional bodies and $n = 1$ for axi-symmetric bodies. $r(x)$ is the radius of the axi-symmetric body. The boundary conditions are

$$v = 0,\ T = T_w \text{ on } y = 0,\ u \to 0,\ T \to T_0 \text{ as } y \to \infty. \tag{4}$$

Using l as a typical length scale, equations (1), (2) and (3)

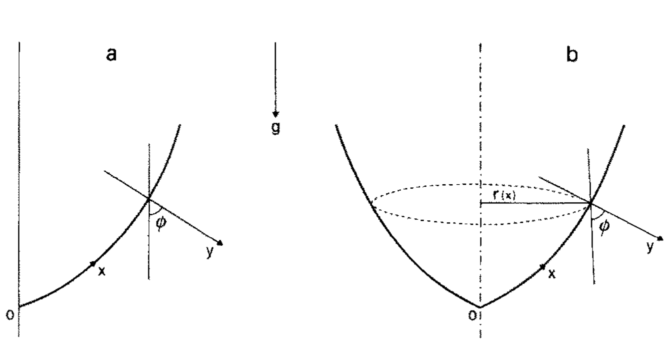

FIG. 1. Co-ordinate systems: (a) two-dimensional bodies; (b) axi-symmetric bodies.

can be made non-dimensional by putting

$$T-T_0 = \Delta T\theta,\ u = \left(\frac{g\beta K\rho\Delta T}{\mu}\right)U,\ v = \left(\alpha\frac{g\beta K\rho\Delta T}{\mu l}\right)^{1/2}V$$

$$x = Xl,\ y = Y\left(\frac{\alpha\mu l}{g\beta K\rho\Delta T}\right)^{1/2} \text{ and } r = Rl.$$

Writing $\sin\phi = S(X)$, this gives

$$U = \theta S(X) \quad (5)$$

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y} = \frac{\partial^2\theta}{\partial Y^2} \quad (6)$$

$$\frac{\partial}{\partial X}(R^nU)+\frac{\partial}{\partial Y}(R^nV) = 0 \quad (7)$$

with boundary conditions

$$V = 0,\ \theta = 1 \text{ on } Y = 0,\ U\to 0,\ \theta\to 0 \text{ as } Y\to\infty. \quad (8)$$

Equations (5)–(7) possess a similarity solution as follows. We define a stream function ψ by $U = (1/R^n)(\partial\psi/\partial Y)$ and $V = -(1/R^n)(\partial\psi/\partial X)$ and then put

$$\theta = \theta(\eta),\ \psi = \left(2\int_0^X R^{2n}(t)S(t)\,\mathrm{d}t\right)^{1/2} f(\eta)$$

$$\eta = YS(X)R^n(X)\Bigg/\left(2\int_0^X R^{2n}(t)S(t)\,\mathrm{d}t\right)^{1/2}.$$

With $S(X) = 1$ and $n = 0$ the above gives $\psi = (2X)^{1/2}f(\eta)$, $\eta = Y/(2X)^{1/2}$ and the similarity solution for the vertical flat plate derived by Cheng and Minkowycz [6] is recovered. Equations (5) and (6) then become

$$f' = \theta \quad (9)$$

$$\theta''+f\theta' = 0 \quad (10)$$

where dashes denote differentiation with respect to η. Using (9), (10) becomes

$$f'''+ff'' = 0 \quad (11)$$

with boundary conditions

$$f(0),\ f'(0) = 1,\ f'\to 0 \text{ as } \eta\to\infty. \quad (12)$$

Equation (11) has been solved previously by Ackroyd [9], from which it is found that $f''(0) = -0.62756$. The local heat transfer $Q = -k(\partial T/\partial y)_{y=0}$ is then

$$Q = 0.62756\frac{S(X)R^n(X)}{\left(2\int_0^X S(t)R^{2n}(t)\,\mathrm{d}t\right)^{1/2}} \times k\Delta T\left(\frac{g\beta K\rho\Delta T}{\alpha\mu l}\right)^{1/2}.$$

REFERENCES

1. W. J. Minkowycz and P. Cheng, Free convection about a vertical cylinder embedded in a porous medium, *Int. J. Heat Mass Transfer* **19**, 805 (1976).
2. P. Cheng and I. D. Chang, Buoyancy induced flows in a saturated porous medium adjacent to impermeable horizontal surfaces, *Int. J. Heat Mass Transfer* **19**, 1267 (1976).
3. P. Cheng, The influence of lateral mass flux on free convection boundary layers in a saturated porous medium, *Int. J. Heat Mass Transfer* **20**, 206 (1977).
4. P. Cheng, Combined free and forced convection flow about inclined surfaces in porous media, *Int. J. Heat Mass Transfer* **20**, 807 (1977).
5. P. Cheng, Similarity solutions for mixed convection from horizontal impermeable surfaces in saturated porous media, *Int. J. Heat Mass Transfer* **20**, 893 (1977).
6. P. Cheng and W. J. Minkowycz, Free convection about a vertical flat plated embedded in a porous medium with application to heat transfer from a dike, *J. Geophys. Res.* **82**, 2040 (1977).
7. J. H. Merkin, Free convection boundary layers in a saturated porous medium with lateral mass flux, *Int. J. Heat Mass Transfer* **21**, 1499 (1978).
8. C. H. Johnson and P. Cheng, Possible similarity solutions for free convection boundary layers adjacent to flat plates in porous media, *Int. J. Heat Mass Transfer* **21**, 709 (1978).
9. J. A. D. Ackroyd, On the laminar compressible boundary layer with stationary origin on a moving flat wall, *Proc. Camb. Phil. Soc.* **63**, 871 (1967).
10. W. H. Braun, S. Ostrach and J. E. Heighway, Free convection similarity flows about two-dimensional and axi-symmetric bodies with closed lower ends, *Int. J. Heat Mass Transfer* **2**, 121 (1961).
11. R. A. Wooding, Convection in a saturated porous medium at large Rayleigh number of Peclet number, *J. Fluid Mech.* **15**, 527 (1963).